MATHEMATICAL FEATURES OF INTEGRATED TIME SCALE AND STATISTICS OF FREQUENCY FLUCTUATIONS

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RIASSUNTO. — Sono studiate le proprietà statistiche ed i caratteri sistematici delle fluttuazioni di frequenza di oscillatori a quarzo dedotte da confronti di fase fra la frequenza dell'oscillatore locale e quella trasmessa dalla stazione *GBR*. Viene messa in evidenza una probabile correlazione tra la frequenza istantanea e la variazione di frequenza di un oscillatore. Un'analisi preliminare del rumore mette in evidenza l'esistenza di fluttuazioni di tipo flicker noise che possono essere attribuite a fluttuazioni causate dalla propagazione, come rilevato da ALLAN e BARNES (1967). In questa ipotesi le fluttuazioni di base, cioè il rumore di tipo « random walk » possono essere invece attribuite all'orologio fondamentale.

SUMMARY. — Non systematic and statistical properties of the frequency fluctuations of a quartz clock compared with the *GBR* station have been examined. A reasonable correlation between the instantaneous frequency and the frequency rate is emphasized. A preliminary analysis of the noise shows the existence of flick noise phase modulation very likely caused by propagation phenomena. The base fluctuations of random walk of phase noise type can be attributed on the contrary to the master clock.

1. - INTRODUCTION

The fundamental expression for the instantaneous frequency of a quartz or atomic oscillator can be written

$$\Omega(t) = \Omega_0 \left[1 + \alpha(t) + \varepsilon(t)\right]. \tag{1}$$

Where $\alpha(t)\,\Omega_0$ represents the instantaneous frequency rate and $\varepsilon(t)$ is a frequency variation depending upon the noise present in the oscillator itself.

The average angular frequency

$$\Omega(T) = \frac{1}{T}\int_{-T/2}^{+T/2} \Omega(t)\, at$$

(*) Ricevuta l'11 Maggio 1970. Testo definitivo il 30 Luglio 1970.

determined for the sampling period T can be written through eq. (1) in the following form

$$\Omega(T) = \Omega_o + \frac{\Omega_o}{T} \int_{-T/2}^{+T/2} \left[\alpha(t) + \varepsilon(t)\right] dt \tag{1.1}$$

from which the time scale can be deduced by means of an expression of the form

$$\Phi(T) = T\,\Omega(T) = T\,\Omega_o \left[1 + \frac{\alpha}{2} T + \int_{-T/2}^{+T/2} \varepsilon(t)\, dt\right] \tag{2}$$

in the case that

$$\alpha(t) = \alpha\, t\ . \tag{3}$$

This last relation nevertheless is true only in a first approximation.

The deviations of the real trend of a quartz oscillator from the relation (3) can be studied analysing the behaviour of frequency variations $\varepsilon(t)$ in relation (2) with the theoretical time scale

$$\overline{\Phi}(T) = T\,\Omega_o \left(1 + \frac{\alpha}{2} T\right). \tag{2.1}$$

2. - Statistical correlations

The study of the differences $\Phi(T) - \overline{\Phi}(T)$ of a quartz oscillator Ebauches type P 31, deduced from comparison with *GBR* radio station as received at the Milan Observatory, have put in evidence a characteristic trend of these differences. In Fig. 1 is represented the behaviour of frequency deviation's for two typical cases

$$\left(\frac{\Delta_1 f}{f}\right)_{obs} - \left(\frac{\Delta_1 f}{f}\right)_{cal} = \frac{d}{dt}\left[\Phi(T) - \overline{\Phi}(T)\right].$$

These differences are calculated for periods of the order of ten days, included between two successive frequency corrections applied to the master oscillator. In Fig. 1 the black dots are characteristic of the oscillator trend for a frequency range characterised by the expression

$$\Omega(T) \leq \Omega_o\ , \tag{4.1}$$

namely in the range of frequency lower than the quartz conventional frequency Ω_o. On the contrary dots are referred to the oscillator trend for a range of frequency

$$\Omega(T) \geq \Omega_o. \tag{4.2}$$

The explanation of this fact seems to be connected to an apparent correlation between the instantaneous frequency and the frequency rate $\alpha\Omega_o = \Delta_2 f/f =$

$d^2 \Phi (T)/dt^2$ of the oscillator. This correlation is put in evidence in Fig. 2 in which the arrows represent the real increase with time of the instantaneous frequency. In Fig. 2 the full dots related to values of the quantities $\Delta_1 f/f$ and $\Delta_2 f/f$ are determined in the frequency range assumed in the relation (4.1). The white dots are related to frequency range which follows the relation (4.2).

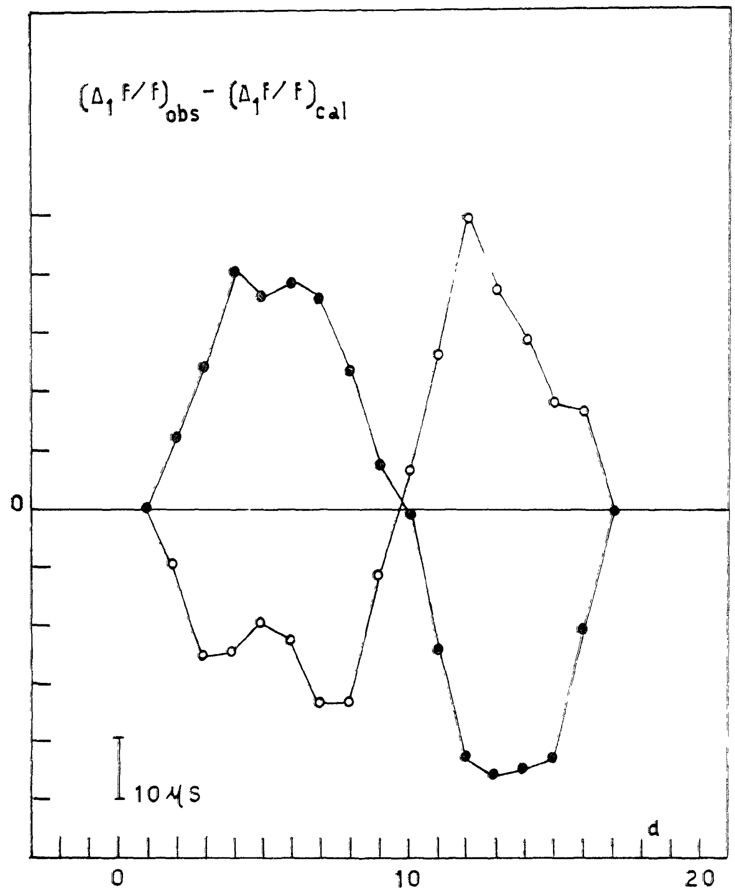

FIG. 1

Typical case of frequency deviations behaviour of a quartz clock. Black dots show the oscillator trend for frequencies smaller of the conventional frequency Ω_o of the oscillator. White dots are referred to frequencies greater than Ω_o.

3.1 - Analysis of noise processes

The behaviour of the frequency fluctuations described in the foregoing section can be studied also analysing the frequency variations with the method of the statistical analysis which allows to give additional information on the noise characteristics of the quartz oscillator. In fact when dealing with precise frequency measurements and time keeping, the properties and characteristics of noise play an important role. The study of these characteristics has been carried out by computing the standard deviation $\sigma(N, T)$, depending on the number of samples N and of the period of sampling T, which, in our case, results constant and equal to the frequency measurement interval τ (24 hours).

The standard deviation, calculated by means of the relation

$$\sigma^2(N,T) = \frac{1}{N-1}\left\{\sum_{n=0}^{N-1}\left[\frac{\Phi(nT+T)-\Phi(nT)}{T}\right]^2 - \frac{1}{N}\left[\sum_{n=0}^{N-1}\frac{\Phi(nT+T)-\Phi(nT)}{T}\right]^2\right\}$$

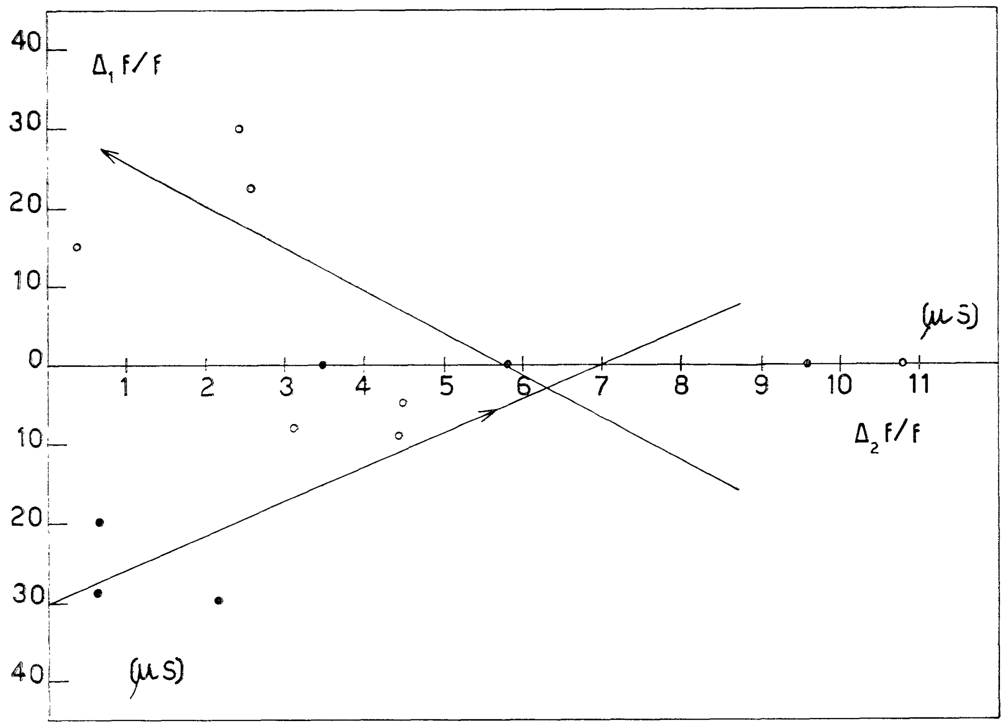

FIG. 2

Empirical correlation between the instantaneous frequency and the frequency rate of the quartz oscillator.

permits, according to ALLAN (1966), to utilize this quantity for studying the statistical properties of quartz standard frequencies. Considering the phase fluctuations like random fluctuations the variance $\sigma^2(N, T)$ if the ratio T/τ is held constant along whith N, may be written (ALLAN e BARNES 1967)

$$\sigma^2(N,T) = a(\mu)\,|\,\tau\,|^{\mu} \tag{5}$$

where $a(\mu)$ is a function of N, T and μ are related to the parameter α characterizing the spectral density of the random fluctuations as shown in Table I.

TABLE I

α	μ	noise type
-2	1	Random walk noise
-1	0	Flicker noise
0	-1	White noise
$+1$	-2	White noise
$+2$	-2	White noise

The calculation of the standard deviation in function of N appears very efficacious for the determination of some types of noise, particularly for the so called *flicker noise*, defined as a random noise whose spectral density is inversely proportional to the spectral frequency. In fact the ratio

$$(6) \qquad \chi(N,\alpha) = \frac{\sigma^2(N, T)}{\sigma^2(2, T)}$$

results a function only of N and of the spectral type of noise characterised by the parameters α or μ.

3.2 - The values of the quantities $\sigma_2^2(N, T)$ in function of N have been calculated for a deviation $\Phi(T) - \overline{\Phi}(T)$ relative to different values of N, that is for more or less long periods, supposing that the calculated constant α, given by (3), were variable with N. In Fig. 3 is represented the plot of these quantities. On the contrary, utilizing for the determination of $\Phi(T) - \overline{\Phi}(T)$ the same value of the parameter α for each series of observations and varying N, one obtains proportionally larger values of $\sigma_1^2(N, T)$ as it is shown in Fig. 4.

The ratio σ_1^2/σ_2^2 can be expressed with the following experimental relation

$$\frac{\sigma^2_1}{\sigma^2_2} \simeq 1 + \left(\frac{N}{12} + 1\right)^{-1} ,$$

while one can write

$$(7.1) \qquad \sigma^2_2 \simeq 30 + 2{,}5\,N \, , \qquad (\mu s)$$

$$(7.2) \qquad \sigma^2_1 \simeq 60 + 2{,}5\,N \, . \qquad (\mu s)$$

the quantity $\sigma^2 = \pm\ 32.5$ μs represents therefore the values of the standard deviation corresponding to a sampling period of 24 hours. The values of the quantity $\chi(N, \alpha)$ calculated by means of relation (7.1) and (7.2) are summarised in Table II. The comparison between these data and those calculated theoretically by BARNES (1967) gives the μ values written in the last column of this last table.

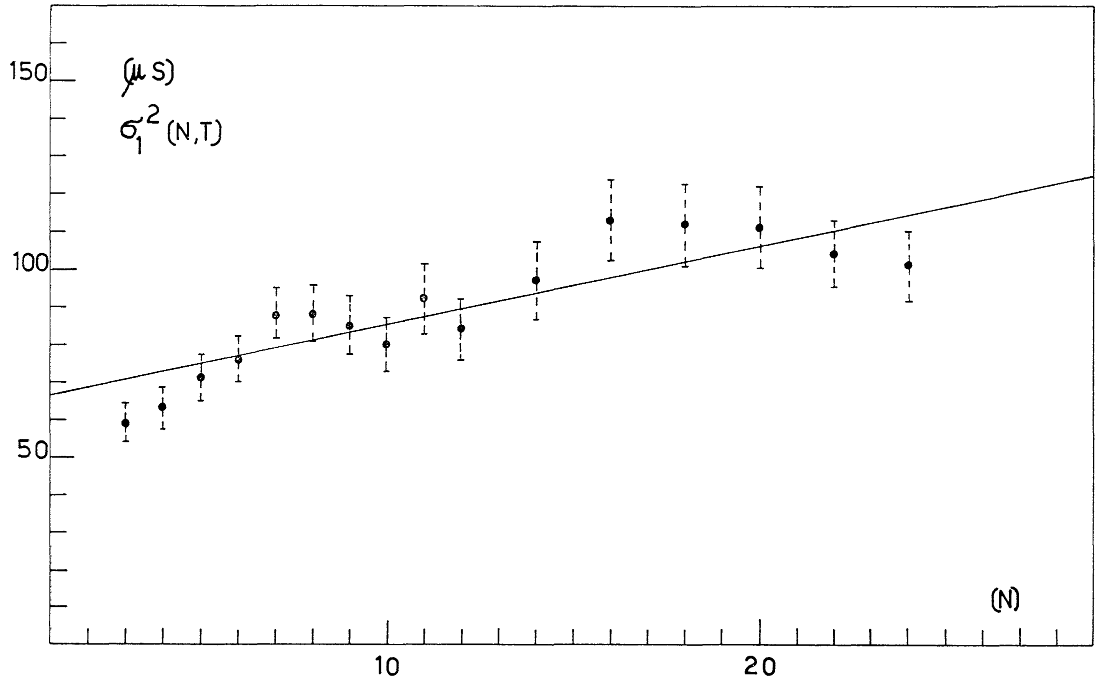

FIG. 3

Plot of the values of the variance σ_2^2 (N, T) of the frequency fluctuations vs. the number N of samples (days).

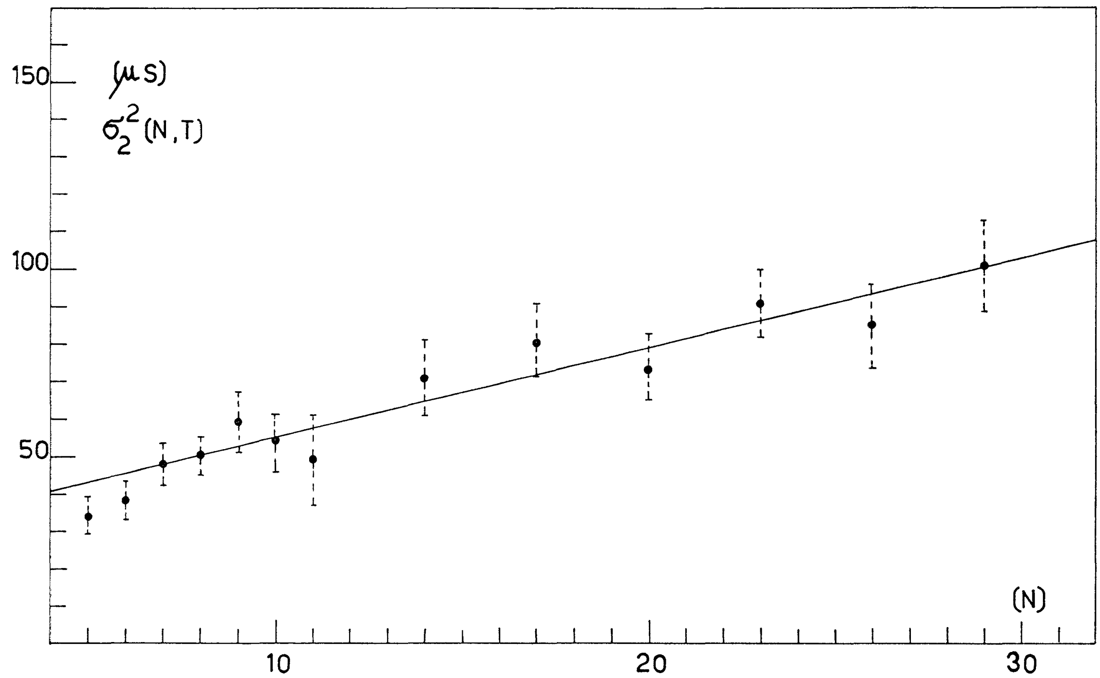

FIG. 4

Plot of the values of the variance σ_2^2 (N, T) of the frequency fluctuations vs. N considering the rate parameter a constant varying N.

TABLE II

	$\chi(4, \alpha)$	$\chi(8, \alpha)$	$\chi(16, \alpha)$	μ
δ^2_1	1.14	1.43	1.71	-0.3
δ^2_2	1.08	1.23	1.38	-0.5
δ^2	2.00	4.00	8.00	$+1.0$

The noise type characteristic for frequency variation in a quartz oscillator results therefore of an intermediate type between white and flicker noise. Disregarding the base fluctuation characteristic of frequency comparisons independent of the number N of sampling, one obtains finally the $\chi(N, \alpha)$ values written in the last line of Table II. The corresponding value of the μ parameter shows that the noise is of the random walk type.

4. - Conclusions

The characteristics of the frequency fluctuations of quartz oscillators, resulting from phase comparisons with the *GBR* station, indicates the presence of *flicker noise* modulation. It is not easy to determine the source of this noise and if this type of noise will arise from quartz instabilities or from frequency modulation of the *GBR* station caused by propagation phenomena. We can only observe that the flicker noise spectrum can be attributed to the propagation fluctuation as found by Allan and Barnes (1967) and Guetrot (1969). On this hypothesis, the base fluctuations of random walk noise type can be attributed to the master clock. This type of noise is more divergent because it has been shown that the spectral density results proportional to $|\omega|^{-2}$, nevertheless one can admit that the precision in the conservation and synchronisation of time can be maintained between $\pm$ 10 μs on monthly periods ($N = 30$).

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